Appendix: Demographic Change, Human Capital and Welfare

Alexander Ludwig* Thomas Schelkle† Edgar Vogel‡

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Overview

This appendix of our paper, "Demographic Change, Human Capital and Welfare", contains further material that could not be included in the paper due to space limitations. It is organized as follows. Section A contains the formal equilibrium definition. Section B provides more results on the fit of our model to observed life-cycle profiles of hours and wages, the implied labor-supply elasticities of our model, additional results on predicted aggregate variables during the demographic transition as well as the associated welfare effects and a sensitivity analysis. Our population model is explained in Section C. Details on our computational procedures can be found in Section D.

A Equilibrium

Denoting current period/age variables by x and following period/age variables by x', a household of age j solves the maximization problem at the beginning of period t

$$V(a,h,t,j) = \max_{c,\ell,e,a',h',s'} \{u(c, 1-\ell-e) + \varphi\beta V(a',h',s',t+1,j+1)\} \tag{1}$$

subject to $w^n_{t,j} = \ell_{t,j} h_{t,j} w_t (1-\tau_t)$,

$$a_{t+1,j+1} = \begin{cases} (a_{t,j} + tr_t)(1+r_t) + w^n_{t,j} - c_{t,j} & \text{if } j < jr \\ (a_{t,j} + tr_t)(1+r_t) + p_{t,j} - c_{t,j} & \text{if } j \geq jr, \end{cases} \tag{2}$$

$$h_{t+1,j+1} = h_{t,j}(1-\delta^h) + \xi(h_{t,j}e_{t,j})^\psi \quad \psi \in (0,1), \;\; \xi > 0, \;\; \delta^h \geq 0, \tag{3}$$

and the constraints $\ell \in [0,1)$, $e \in [0,1)$.

*CMR, Department of Economics and Social Sciences, University of Cologne; Albertus-Magnus-Platz; 50923 Cologne; Email: ludwig@wiso.uni-koeln.de.

†London School of Economics (LSE); Houghton Street; London WC2A 2AE; United Kingdom; Email: t.schelkle@lse.ac.uk.

‡Mannheim Research Institute for the Economics of Aging (MEA); Universität Mannheim; L13, 17; 68131 Mannheim; Email: vogel.edgar@utanet.at.

Definition 1. *Given the exogenous population distribution and survival rates in all periods* $\{\{N_{t,j},\varphi_{t,j}\}_{j=0}^{J}\}_{t=0}^{T}$, *an initial physical capital stock and an initial level of average human capital,* $\{K_0,\bar{h}_0\}$, *and an initial distribution of assets and human capital,* $\{a_{t,0},h_{t,0}\}_{j=0}^{J}$, *a competitive equilibrium of the economy is defined as a sequence of individual variables* $\{\{c_{t,j},\ell_{t,j},e_{t,j},a_{t+1,j+1},h_{t+1,j+1},s_{t+1,j+1}\}_{j=0}^{J}\}_{t=0}^{T}$, *sequences of aggregate variables* $\{L_t,K_{t+1},Y_t\}_{t=0}^{T}$, *government policies* $\{\rho_t,\tau_t\}_{t=0}^{T}$, *prices* $\{w_t,r_t\}_{t=0}^{T}$ *and transfers* $\{tr_t\}_{t=0}^{T}$ *such that*

1. *given prices, bequests and initial conditions, households solve their maximization problem, as described above,*
2. *interest rates and wages are paid their marginal products, i.e.,* $w_t=(1-\alpha)\frac{Y_t}{L_t}$ *and* $r_t=\alpha\frac{Y_t}{K_t}-\delta$,
3. *per-capita transfers are determined by*

$$tr_t=\frac{\sum_{j=0}^{J}a_{t,j}(1-\varphi_{t-1,j-1})N_{t-1,j-1}}{\sum_{j=0}^{J}N_{t,j}}, \tag{4}$$

4. *government policies are such that the budget of the social-security system is balanced every period, i.e., Equation (4) in our main text holds* $\forall t$, *and household pension income is given by* $p_{t,j}=\rho_t w_{t+jr-j}\bar{h}_{t+jr-j}\frac{s_{t,j}}{jr-1}$,
5. *markets clear every period:*

$$L_t=\sum_{j=0}^{jr-1}\ell_{t,j}h_{t,j}N_{t,j} \tag{5a}$$

$$K_{t+1}=\sum_{j=0}^{J}a_{t+1,j+1}N_{t,j} \tag{5b}$$

$$Y_t=\sum_{j=0}^{J}c_{t,j}N_{t,j}+K_{t+1}-(1-\delta)K_t. \tag{5c}$$

Definition 2. *A stationary equilibrium is a competitive equilibrium at which per-capita variables grow at the constant rate of* $1+\bar{g}^A$ *and aggregate variables grow at the constant rate* $(1+\bar{g}^A)(1+n)$.

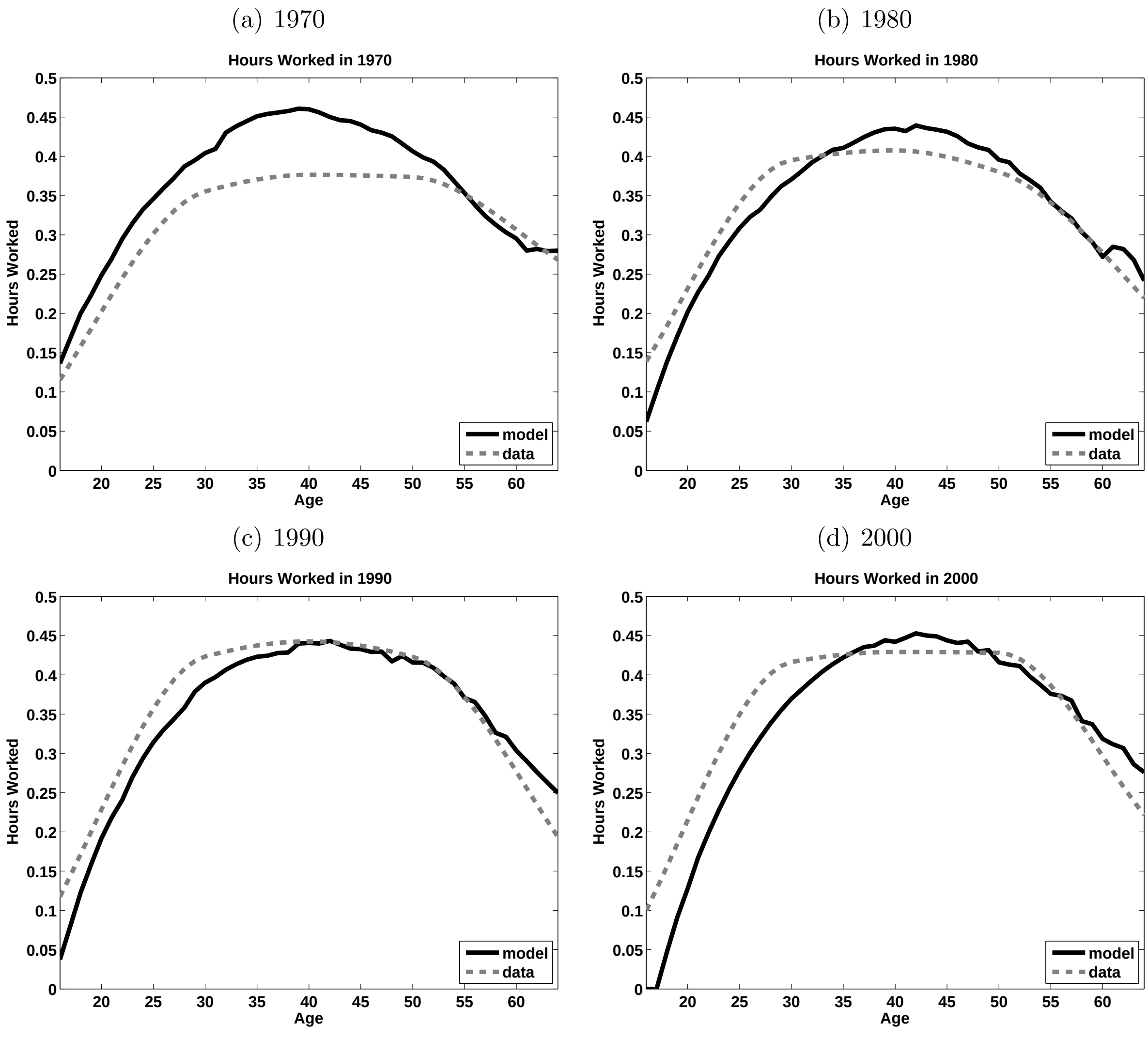
B Further Results

B.1 Backfitting

Figure 1 presents the fit of our model to cross-sectional hours data from McGrattan and Rogerson (2004) for the years 1970, 1980, 1990 and 2000. We observe that our model does a very good job of matching the data along this dimension from 1980 onwards.

A comparison between wage profiles observed in PSID data and the model is shown in Figure 2. The fit of our model is very good in 1970 and 1980 and still broadly consistent with the data in 1990 and 2000.

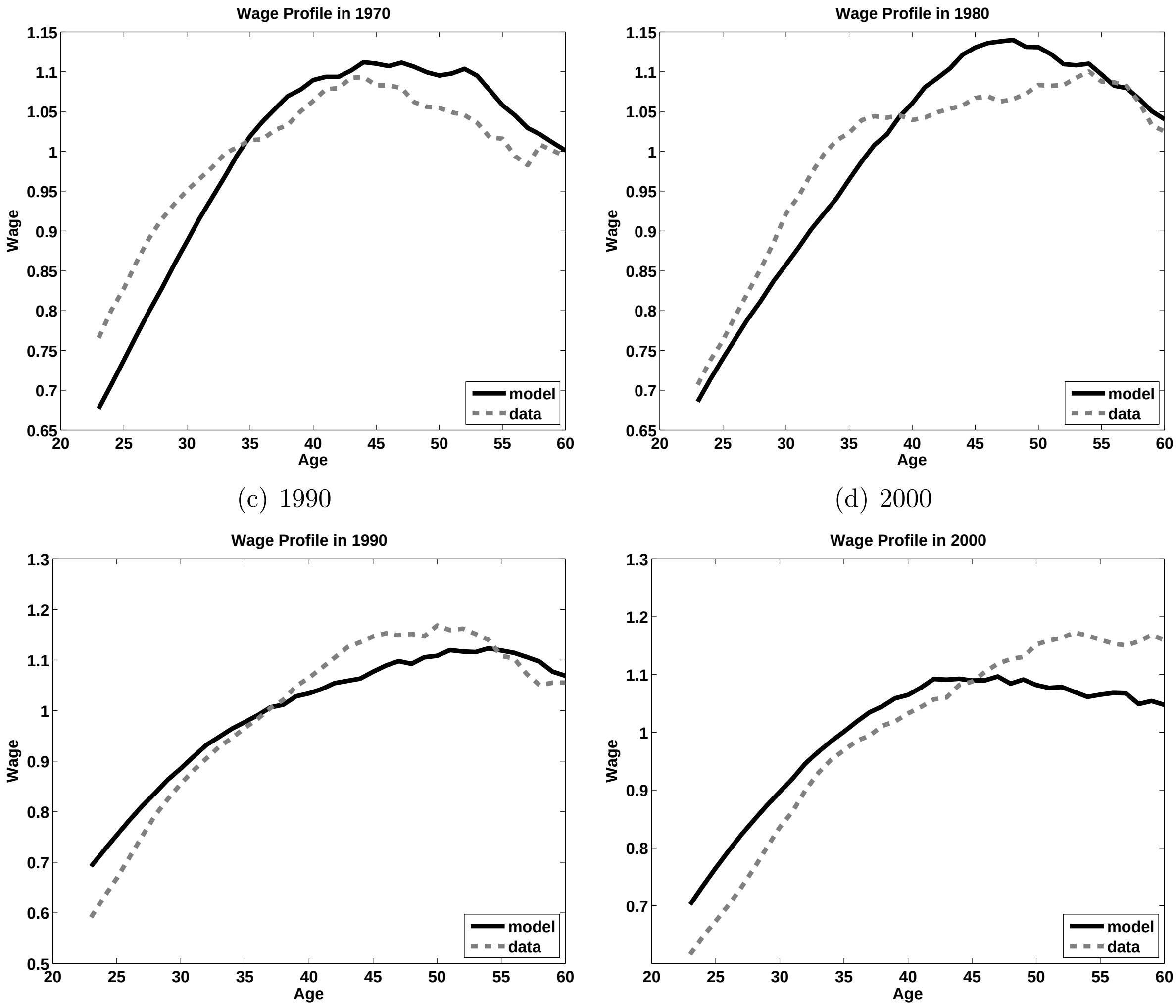
Figure 1: Labor Supply



Notes: Model and data profiles for labor supply. Hours data are normalized by 76 total hours.
Data Sources: Based on hours worked data from the Decennial Censuses obtained from McGrattan and Rogerson (2004).

Figure 2: Wages

(a) 1970 (b) 1980 (c) 1990 (d) 2000



Notes: Model and data profiles for wages. The data are given as a centered average of five subsequent PSID samples.
Data Sources: Based on PSID wage data.

B.2 Labor-Supply Elasticities

Because agents' human-capital investments do not only depend on changes in relative returns but also on the extent of labor-supply adjustments, realistic labor-supply elasticities are key for our analysis. First, we compute the Frisch (or λ-constant) elasticity of labor supply that holds the marginal utility of wealth constant. We do so using the standard formula. In the context of our model, this means holding time invested in human-capital formation constant. It is then given by

$$\epsilon_{\ell,w}^{j} = \frac{1-\phi(1-\sigma)}{\sigma}\frac{1-\ell_j-e_j}{\ell_j}, \tag{6}$$

see Browning, Hansen, and Heckman (1999) for a derivation. In our model, the Frisch elasticity depends on the amount of leisure and labor supply and therefore is age-dependent. As a consequence of the hump-shaped labor supply, the Frisch labor-supply elasticity is u-shaped over the life-cycle. During the years 1960-1995, we find that agents between ages 25 and 50 have a labor-supply elasticity between 0.8 and 1.0, while it is higher for younger and older agents. For agents aged 30-50 (20-60), the average Frisch elasticity is approximately 0.8 (1.0), while across all agents, the average is approximately 1.3. If we aggregate the u-shaped micro-Frisch elasticities to an hours-weighted ("macro") Frisch elasticity,

We also report a Frisch labor-supply elasticity that allows time invested in human-capital formation to vary. In the spirit of the Frisch elasticity concept, we hold the marginal utility of human capital constant in addition to the marginal utility of wealth. This Frisch elasticity is then given by

$$\widetilde{\epsilon}_{\ell,w}^{j} = \frac{1-\phi(1-\sigma)}{\sigma}\frac{1-\ell_j-e_j}{\ell_j} + \frac{1}{1-\psi}\frac{e_j}{\ell_j}. \tag{7}$$

As usual, an interior solution is assumed here. If we use this concept, then the labor-supply elasticity is higher because the second term is positive, i.e., agents invest less in human-capital formation when they face a higher wage today, and the marginal utility of human capital remains unchanged. Due to decreasing time invested in human-capital formation, the second term decreases over the life-cycle. The resulting labor-supply elasticity is still u-shaped over the life-cycle. Accordingly, during 1960-1995 for agents aged 30-50 (20-60), the resulting average Frisch elasticity with varying time investments is around 1.3 (1.8), while across all agents, the average is approximately 2.8. Here, the macro-Frisch elasticity is approximately 1.9 when accounting for the differing initial labor supply across agents of different ages.

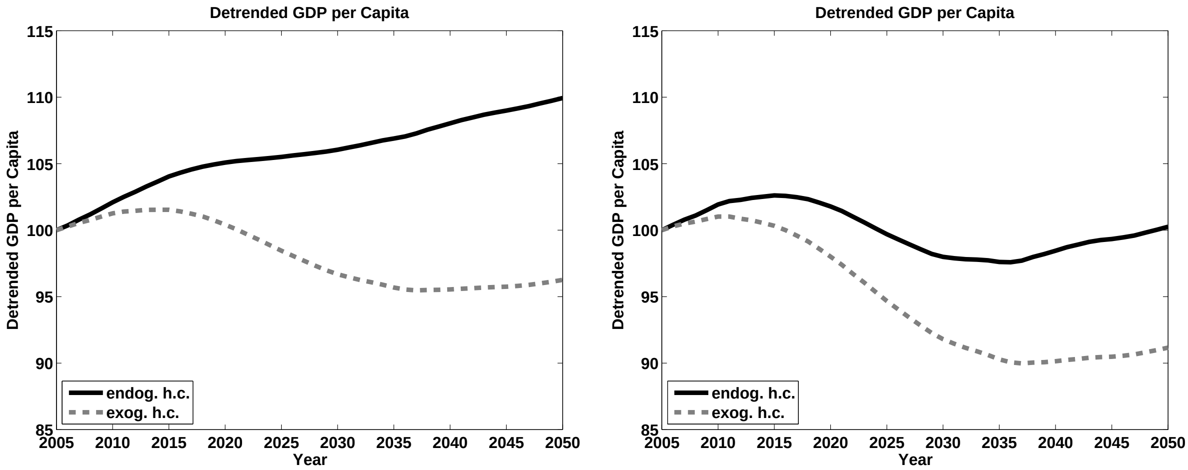
B.3 Transitional Dynamics

B.3.1 Aggregate Variables

The cumulative effect of the differences in growth rates on GDP per capita are displayed in Figure 3. In the endogenous human-capital model with constant contribution (replacement) rates, GDP per capita will increase by approximately 14% (10%) more until the year 2050 than it would without human-capital adjustments.

Figure 3: Detrended GDP per Capita [Index, 2005=100]

(a) Constant Contribution-Rate Scenario (b) Constant Replacement-Rate Scenario



B.3.2 Welfare Effects

Welfare of Future Generations

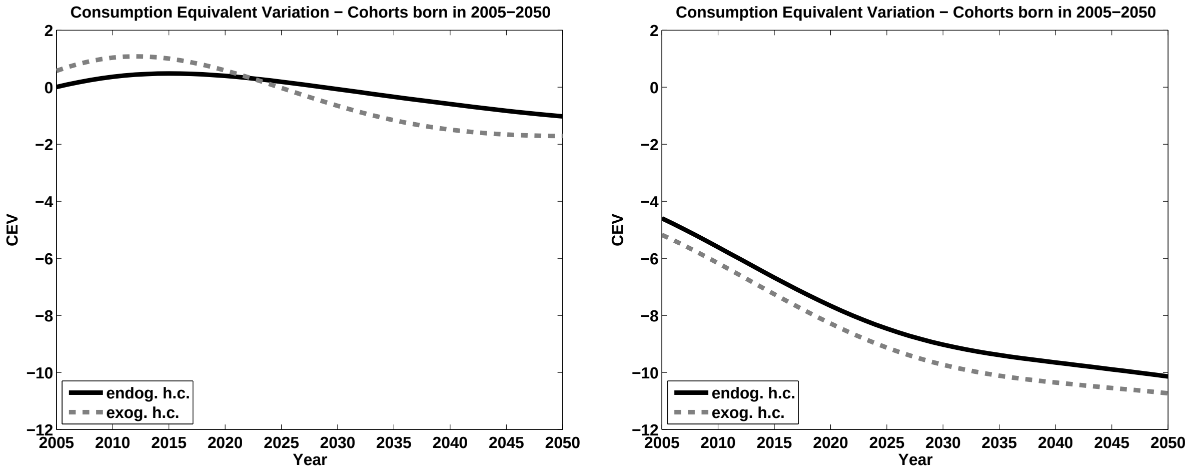
In our main text, we mostly analyze the welfare consequences for agents alive in 2005 and only briefly glance at the consequences for future generations. We here examine those. Figure 4 shows the consumption-equivalent variation for the two models and the two social-security scenarios. Agents born into a "const τ"-world experience welfare gains of up to 1.1% and losses of up to 1.7% of lifetime consumption, depending on whether they are born before or after 2005. Even if agents are allowed to invest in human capital, welfare losses of future generations can be quite large if the contribution rates rise ("const ρ"). Notice, again, that in our comparison across models, differences are not large because the positive value of human-capital adjustments is offset by the more beneficial general- equilibrium effects in the exogenous human-capital model. For this reason, welfare gains for some cohorts may even be slightly higher in the exogenous human-capital model when the contribution rate is held constant.

The Value of Human-Capital Adjustments

From Figure 7 of our main text, we observe that welfare gains (and losses) for newborns are almost identical in the endogenous and exogenous human-capital models. Detailed numbers are provided in Table 1. The explanation for these similar welfare consequences is as follows. While the value of human-capital adjustments is positive (see below), the increase of wages and the associated decrease of interest rates is much stronger in the exogenous human-capital model. As newborn households generally benefit from the combined effects of increasing wages and decreasing returns, welfare gains from these general-equilibrium effects are higher in the exogenous human-capital model. This explains why the overall welfare consequences for newborns across models do not differ much, despite the fact that the value of human-capital adjustments

Figure 4: Consumption Equivalent Variation of Agents born in 2005-2050

(a) Constant Contribution-Rate Scenario (b) Constant Replacement-Rate Scenario



Notes: Consumption-equivalent variation (CEV) in the two social-security scenarios.

is positive.

Table 1: CEV for Generation Born in 2005 [in %]

	Human Capital	
	Endogenous	Exogenous
Const. τ ($\tau_t = \bar{\tau}$)	-0.1%	0.4%
Const. ρ ($\rho_t = \bar{\rho}$)	-4.4%	-5.0%

Notes: CEV: consumption-equivalent variation.

Our comparison across models does not provide any information about the value of a flexible adjustment of human-capital investments from the individual perspective, that is, about the value of human-capital adjustments within the endogenous human-capital model. To accomplish this, we store from our computation of $\bar{V}_j^{2005}$ (see above) the associated endogenous time-investment profile, $\{e_j^{2005}\}_{j=0}^{J}$. Next, we compute $\bar{V}_{t,j}^{CE}$ as the lifetime utility of agents born at time t, age j facing constant 2005 survival rates, a sequence of equilibrium prices, transfers and contribution (replacement) rates, as documented for the endogenous human-capital model in the previous section, but keep the time-investment profile fixed at $\{e_j^{2005}\}_{j=0}^{J}$. Corresponding to our previous work, we then compute

$$g_{t,j}^{CE} = \left(\frac{\bar{V}_{t,j}^{CE}}{\bar{V}_j^{2005}} \right)^{\frac{1}{\phi(1-\sigma)}} - 1, \tag{8}$$

as the consumption-equivalent variation with constant time-investment decisions. The difference

$g_{t,j} - g_{t,j}^{CE}$ is then our measure of the value of endogenous human capital (where $g_{t,j}$ is the consumption-equivalent variation with flexible time investments, as computed above).[1]

The value of human- capital adjustments is obviously positive and more or less monotonically decreasing with age (because of decreasing time investments over the life-cycle). Furthermore, for all future generations, the value of human-capital adjustments can be expected to increase slightly because of the increasing rate of return to human-capital formation. For the sake of brevity, we do not report these results and confine ourselves to a comparison of the value of human-capital adjustments of newborns in 2005, that is $g_{256,0} - g_{256,0}^{CE}$ across social-security scenarios. As reported in Table 2, the value of human-capital adjustments in the "const. τ" scenario is 0.27% compared to 0.22% in the "const. ρ" scenario.

Table 2: The Value of Human-Capital Adjustments in 2005

Const. τ ($\tau_t = \bar{\tau}$)	0.27%
Const. ρ ($\rho_t = \bar{\rho}$)	0.22%

Notes: The value of human-capital adjustments is computed as $g_{t,j} - g_{t,j}^{CE}$.

Role of the Pension System: Agents Alive in 2005

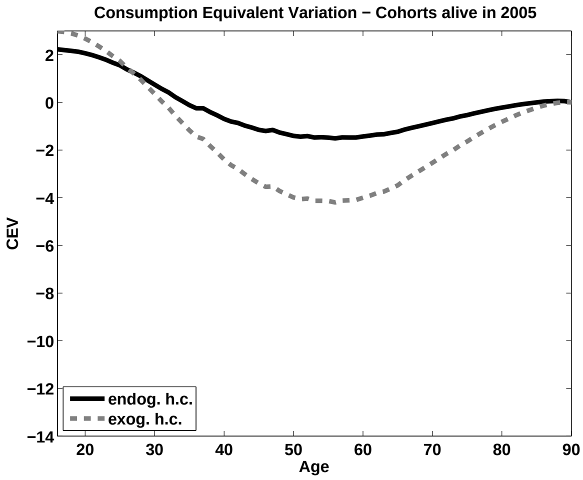
We here provide a decomposition of our welfare results into the effects stemming from changes in relative factor prices and transfers and those of changing pension payments. To this end, Figure 5 shows the welfare consequences of demographic change for agents alive in 2005 from changing factor prices alone, keeping pension payments constant. We here examine only our scenario with constant contribution rates. Table 3 presents the maximum utility loss for agents alive in 2005 with constant pension payments. In the exogenous human-capital model, the maximum loss is approximately 2.7 percentage points, or almost 3 times higher than in the endogenous human-capital model. Observe from Table 2 of our main text that, in terms of the percentage point difference, this gain relative to the exogenous human-capital model is roughly 3.8 percentage points when pension payments adjust. By comparing these numbers, we can therefore conclude that roughly two-thirds of the overall gain of 3.8 percentage points can be attributed to differential changes in interest rates, wages and accidental bequests, and one-third can be attributed to the relative rise in social-security benefits caused by the additional human-capital formation and the accompanying increase of average wages.

Role of Survival Rates for Welfare Calculations

Thus far, we have computed the welfare effects of demographic change by holding survival rates constant. We here present welfare results for varying survival rates. Figures 6 and 7 present the results of these calculations. Table 4 presents the maximum utility loss for agents alive in

[1] To observe this more clearly, rewrite the welfare difference as $g_{t,j} - g_{t,j}^{CE} = (\bar{V}_j^{2005})^{-\frac{1}{\phi(1-\sigma)}} \left(\bar{V}_{t,j}^{\frac{1}{\phi(1-\sigma)}} - \bar{V}_{t,j}^{CE\,\frac{1}{\phi(1-\sigma)}} \right)$. The difference between the terms in the brackets is only due to the fact that agents are (or are not) allowed to adjust their human capital.

Figure 5: CEV of Agents alive in 2005 with Constant Pensions: Constant Contribution Rates



Notes: Consumption-equivalent variation (CEV) in the constant contribution-rate scenario with constant pension payments. "endog. h.c.": endogenous human-capital model with constant pensions. "exog. h.c.": exogenous human-capital model with constant pensions.

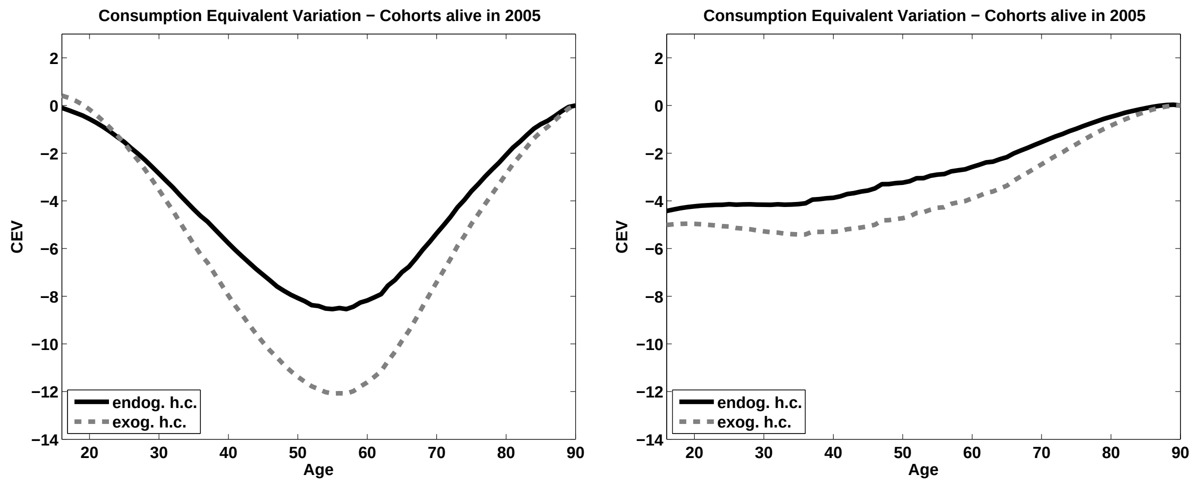
Table 3: Maximum Utility Loss for Generations Alive in 2005 with Constant Pensions

	Human Capital	
	Endogenous	Exogenous
Const. τ ($\tau_t = \bar{\tau}$)	-1.5%	-4.2%

2005 with changing survival rates. Comparing these results to those of Figure 7 and Table 2 in our main text as well as those of Figure 4, we can conclude that holding survival rates constant or varying them according to the underlying demographic projections does not affect our conclusions about the welfare consequences of demographic change in our comparisons across various scenarios.

Figure 6: CEV of Agents Alive in 2005 with changing Survival Rates

(a) Constant Contribution-Rate Scenario (b) Constant Replacement-Rate Scenario



Notes: Consumption-equivalent variation (CEV) calculated with changing survival rates in the two social-security scenarios.

Table 4: Maximum Utility Loss for Generations Alive in 2005 with changing Survival Rates

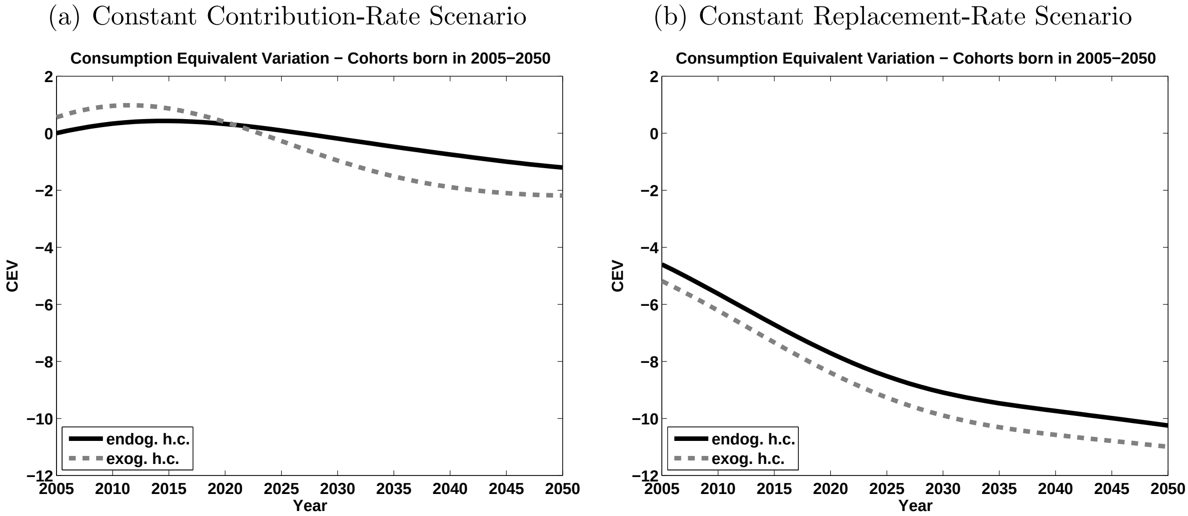
	Human Capital	
	Endogenous	Exogenous
Const. τ ($\tau_t = \bar{\tau}$)	-8.5%	-12.1%
Const. ρ ($\rho_t = \bar{\rho}$)	-4.4%	-5.4%

B.4 Sensitivity Analysis with respect to σ

We now provide a sensitivity analysis with respect to the parameter σ, the inverse of the intertemporal elasticity of substitution. In our benchmark model, we set $\sigma = 2$, but we now also explore the cases of $\sigma = 1$ (log-utility) and $\sigma = 3$. We recalibrate the model when we vary σ, such that we match the same calibration targets as in the main text.

This exercise serves two purposes. First, because σ is a predetermined parameter in our calibration procedure, it is interesting to observe how much our results depend on our choice of σ. Second, we want to investigate how sensitive our results are to changes of the theoretical Frisch

Figure 7: CEV of Agents born in 2005-2050 with changing Survival Rates



Notes: Consumption-equivalent variation (CEV) calculated with changing survival rates in the two social-security scenarios.

labor-supply elasticities in our model. Table 5 shows how varying σ generates variation in these elasticities in the differently calibrated versions of our model.[2] We observe that these experiments generate substantial variation in labor-supply elasticities. With $\sigma = 3$ and a constant (variable) time investment, the "macro" elasticity is approximately 15% (6%) lower than in the benchmark calibration, while with $\sigma = 1$, it is approximately 46% (21%) higher than in the benchmark. A limitation of this sensitivity check is, of course, that we cannot separately identify the effects of the inverse of the inter-temporal elasticity of substitution and the Frisch labor-supply elasticity on our results.

Table 5: Sensitivity Analysis with respect to σ: Mean Frisch Labor-Supply Elasticities during 1960-1995

	Time Investment Constant			Time Investment Variable		
	$\sigma = 1$	$\sigma = 2$	$\sigma = 3$	$\sigma = 1$	$\sigma = 2$	$\sigma = 3$
Age 30 to 50	1.2	0.8	0.7	1.6	1.3	1.2
Age 20 to 60	1.5	1.0	0.9	2.2	1.8	1.7
All Ages	2.0	1.3	1.2	3.3	2.8	3.0
"Macro"	1.6	1.1	0.9	2.3	1.9	1.8

Figure 8 shows that the fit of our model to the observed cross-sectional profiles of consumption, assets, hours and wages is very similar for all values of σ considered here. Unfortunately,

[2] Section B.2 explains how the Frisch labor-supply elasticity depends on σ.

the failure of our model to match the observed consumption profile cannot be fixed by varying σ in this way.

We next turn to the implications of the different parameterizations on the transitional dynamics of the macroeconomic variables. We omit the figures for the contribution and replacement rates of the pension system because the dynamics for the alternative values of σ are basically identical to the benchmark model. The maximum absolute deviation of the contribution rate from the benchmark model in any year is approximately 0.6 percentage points, and for the replacement rate, it is 1.2 percentage points. However, for the vast majority of years, the deviations are much smaller.

Figure 9 presents the evolution of the four major macroeconomic variables for the "const. τ" social-security scenario, and Figure 10 does so for the "const. ρ" scenario. We observe that for the alternative values of σ, the broad dynamics of these variables are very similar to the benchmark model. The most significant differences are that for log-utility ($\sigma = 1$), when human capital is exogenous, average hours increase by more, and the interest rate decreases by less than in the benchmark calibration with $\sigma = 2$ in the years after 2020.

For the different values of σ, the welfare analysis of demographic change for agents alive in 2005 is presented in Figure 11 and Table 6. The welfare results can be viewed as an important and convenient summary measure of all of the differences between differently parameterized models. We find that the welfare assessment of demographic change does not depend much on the value of σ and the comparison across models with endogenous and exogenous human capital is largely unaffected. We thus conclude that our main quantitative result that human-capital adjustments mitigate the macroeconomic and welfare effects of demographic change is robust to the changes of σ we considered here.

Table 6: Sensitivity Analysis with respect to σ: Maximum Utility Loss for Generations Alive in 2005

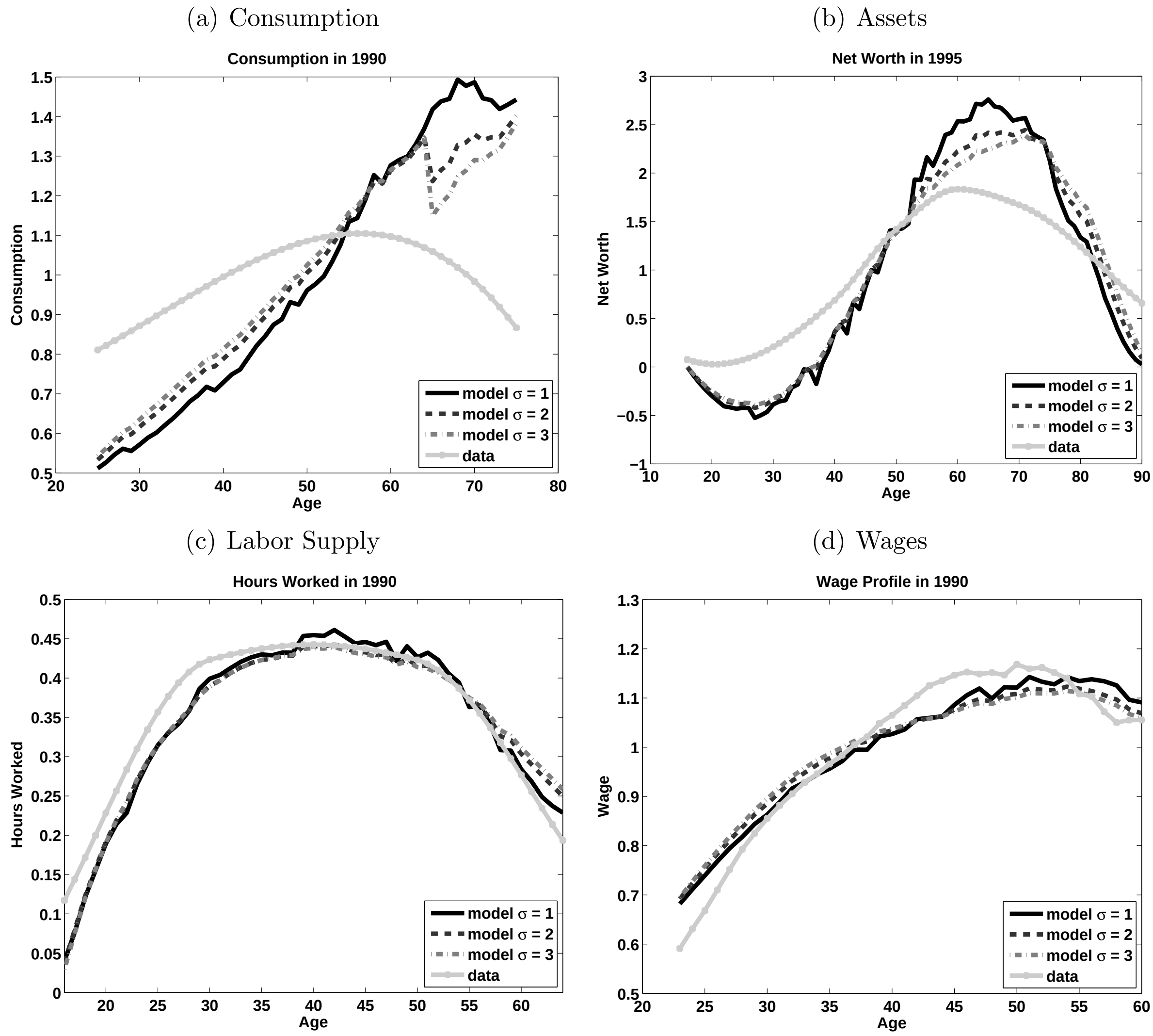
	$\sigma = 1$		$\sigma = 2$		$\sigma = 3$	
	Human Capital		Human Capital		Human Capital	
	Endog.	Exog.	Endog.	Exog.	Endog.	Exog.
Const. τ ($\tau_t = \bar{\tau}$)	-8.4%	-11.4%	-8.7%	-12.5%	-8.7%	-13.3%
Const. ρ ($\rho_t = \bar{\rho}$)	-4.4%	-5.8%	-4.4%	-5.6%	-4.3%	-5.6%

C Demographic Data

Our demographic data are based on the Human Mortality Database (2008). Population of age j in year t is determined by four factors: (i) an initial population distribution in year 0, (ii) age- and time-specific mortality rates, (iii) age- and time-specific fertility rates and (iv) age- and time-specific migration rates. We describe here how we model all of these elements and then briefly compare results of our demographic predictions with those of United Nations (2007).

Figure 8: Sensitivity Analysis with respect to σ: Cross-Sectional Profiles

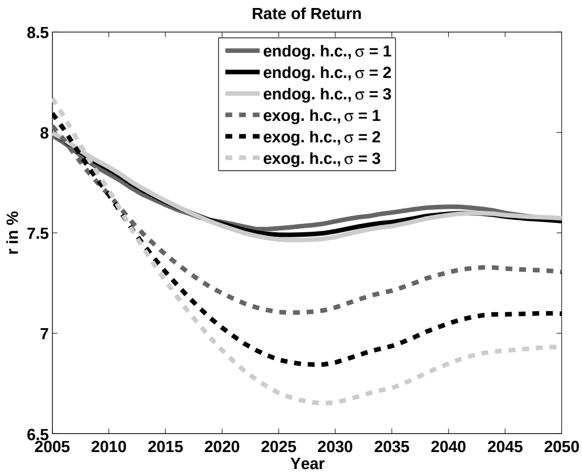
(a) Consumption (b) Assets (c) Labor Supply (d) Wages



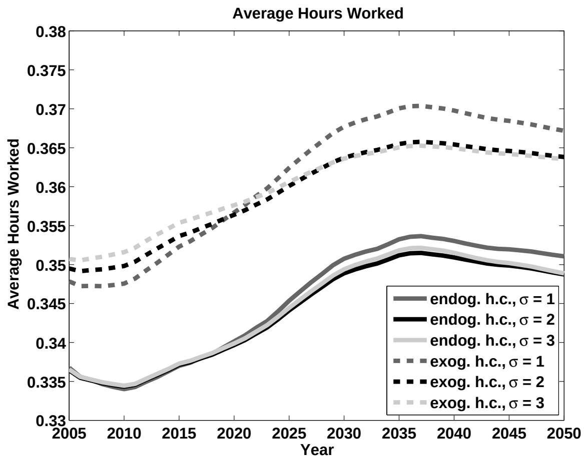
Notes: Model and data profiles for consumption, assets, labor supply and wages. The model profiles are for values of σ equal to 1, 2 (our benchmark value) and 3. All profiles are cross-sectional profiles in 1990, except for the asset profile, which is for 1995. Consumption, asset and wage profiles are normalized by their respective means. Hours data are normalized by 76 total hours per week.
Data Sources: Based on CEX consumption data collected from Aguiar and Hurst (2009), SCF net worth data obtained from Bucks et al. (2006), hours worked data from McGrattan and Rogerson (2004) and PSID wage data.

Figure 9: Sensitivity Analysis with respect to σ: Aggregate Variables for the Constant Contribution-Rate Scenario

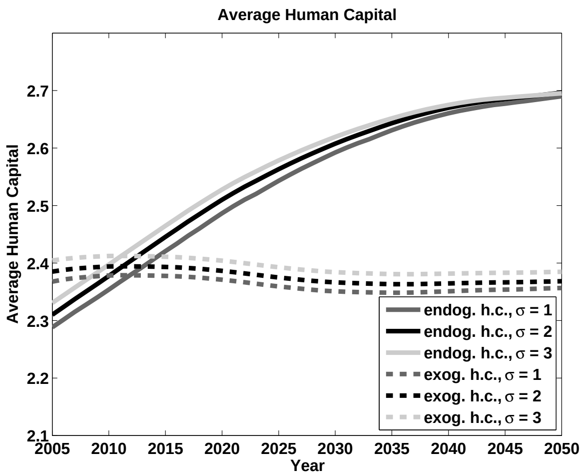
(a) Rate of Return to Physical Capital



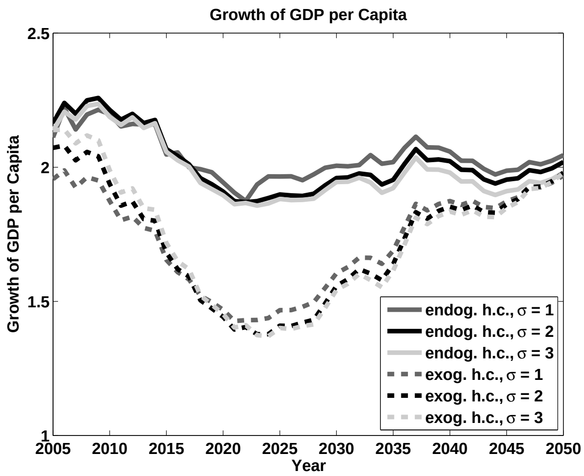
(b) Average Hours Worked



(c) Average Human Capital



(d) Growth of GDP per Capita in %



Notes: Rate of return to physical capital, average hours worked of the working-age population, average human capital per working hour and growth of GDP per capita in the constant contribution-rate social-security scenario for two model variants and three different values of σ. "endog. h.c.": endogenous human-capital model. "exog. h.c.": exogenous human-capital model.

Initial Population Distribution

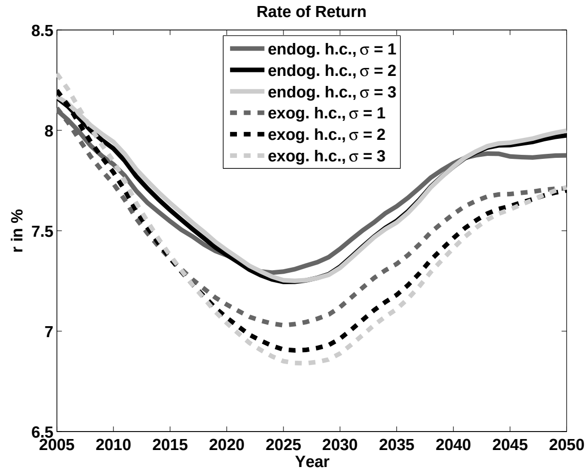
We collect the age- and time-specific population data for the period $1950-2004$.

Mortality Rates

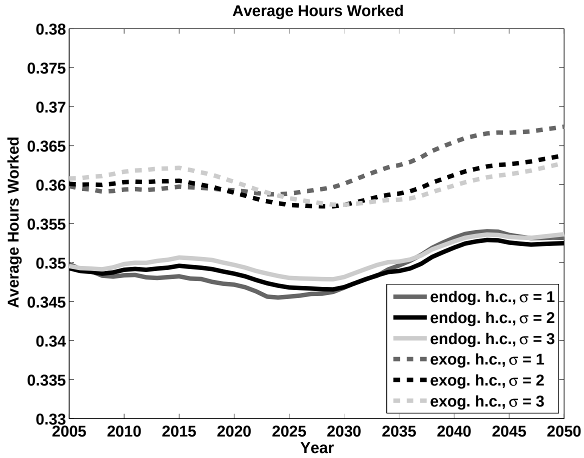
Our mortality model is based on sex-, age- and time-specific mortality rates. To simplify notation, we suppress a separate index for sex. Using data from $1950-2004$, we apply a Lee-Carter

Figure 10: Sensitivity Analysis with respect to σ: Aggregate Variables for the Constant Replacement-Rate Scenario

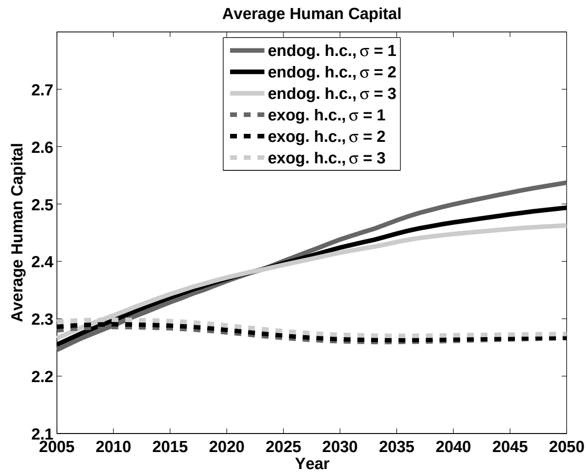
(a) Rate of Return to Physical Capital



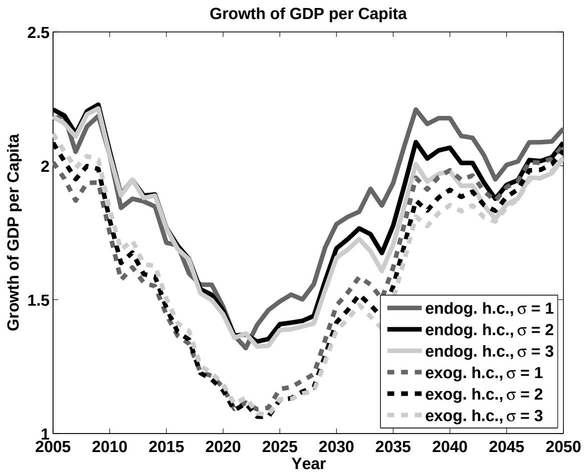
(b) Average Hours Worked



(c) Average Human Capital



(d) Growth of GDP per Capita in %



Notes: Rate of return to physical capital, average hours worked of the working-age population, average human capital per working hour and growth of GDP per capita in the constant replacement-rate social-security scenario for two model variants and three different values of σ. "endog. h.c.": endogenous human-capital model. "exog. h.c.": exogenous human-capital model.

procedure (Lee and Carter 1992) to decompose mortality rates as

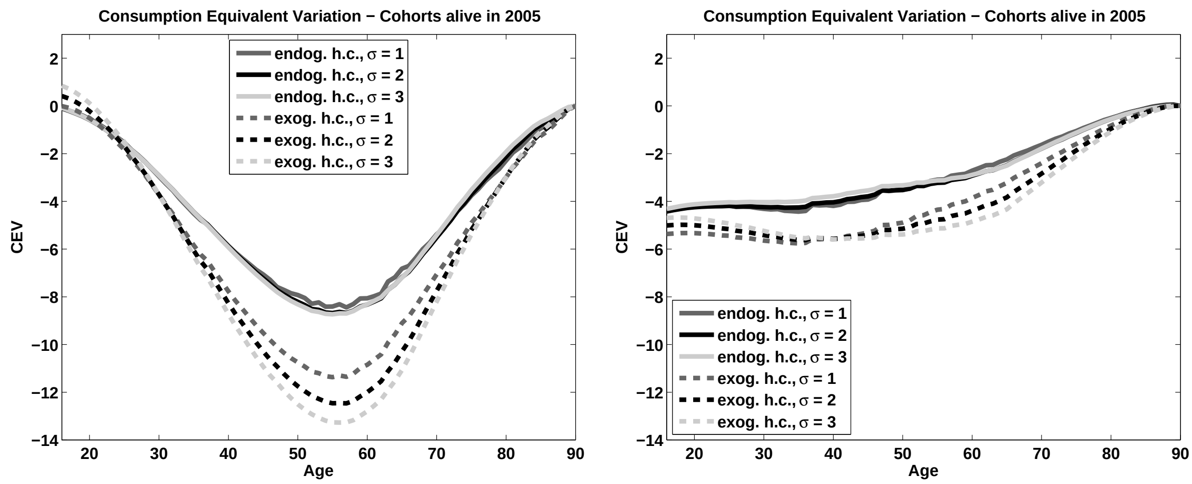
$$\ln(1 - \varphi_{t,j}) = a_j + b_j d_t, \tag{9}$$

where a_j and b_j are vectors of age-specific constants, and d_t is a time-specific index that equally affects all age groups. We assume that the time-specific index, d_t, evolves according to a unit-root process with drift,

$$d_t = \chi + d_{t-1} + \epsilon_t. \tag{10}$$

Figure 11: Sensitivity Analysis with respect to σ: Consumption-Equivalent Variation of Agents Alive in 2005

(a) Constant Contribution-Rate Scenario (b) Constant Replacement-Rate Scenario



Notes: Consumption-equivalent variation (CEV) in the two social-security scenarios for three different values of σ. “endog. h.c.”: endogenous human-capital model. “exog. h.c.”: exogenous human-capital model.

The estimate of the drift term is $\hat{\chi} = -1.2891$. We then predict mortality rates into the future (until 2100) by holding $\hat{a}_j$, $\hat{b}_j$ and $\hat{\chi}$ constant and setting $\epsilon_t = 0$ for all t. For all years beyond 2100, we hold survival rates constant at their respective year 2100 values. Figure 12 shows the corresponding path of life expectancy at birth.

Fertility Rates

Fertility in our model is age and time specific. For our predictions, we assume that age-specific fertility rates are constant at their respective year 2004 values for all periods $2005, \ldots, 2100$. For periods after 2100, we assume that the number of newborns is constant. Because the U.S. reproduction rate is slightly above replacement levels, this implies that the total fertility rate is slightly decreasing each year from 2100 onwards, until approximately year 2200, when the population converges to a stationary distribution.

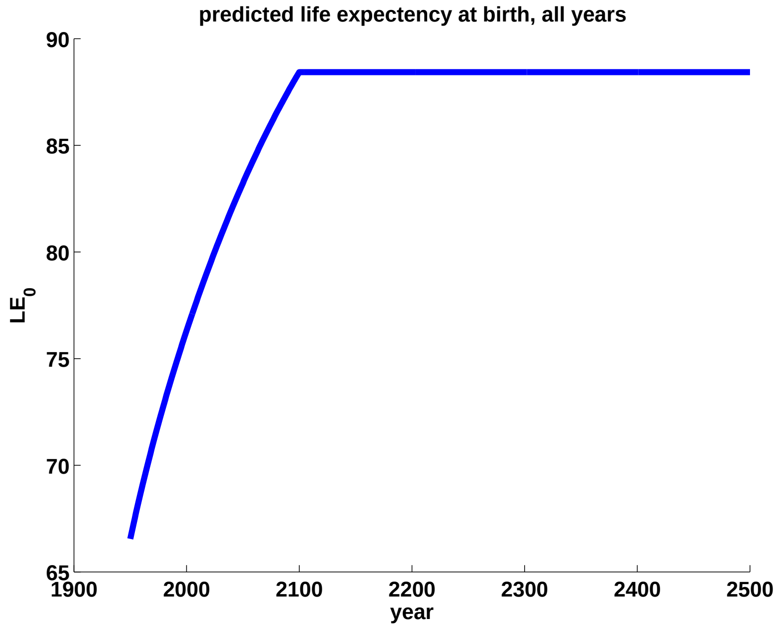
Population Dynamics

We use the estimated fertility and mortality data to forecast future population dynamics. The transition of the population is accordingly given by

$$N_{t,j} = \begin{cases} N_{t-1,j-1}\varphi_{t-1,j-1} & \text{for } j > 0 \\ \sum_{i=0}^{J} f_{t-1,i} N_{t-1,i} & \text{for } j = 0, \end{cases} \tag{11}$$

where $f_{t,j}$ denotes age- and time- specific fertility rates. Population growth is then given by $n_t = \frac{N_{t+1}}{N_t} - 1$, where $N_t = \sum_{j=0}^{J} N_{t,j}$ is total population in t.

Figure 12: Life Expectancy at Birth



Notes: Our own predictions of life expectancy at birth based on Human Mortality Database (2008).

Migration

Migration is exogenous in our economic model. Setting migration equal to zero would lead us to overestimate future decreases in the working-age population ratio and to overstate the increases in old-age dependency. We therefore restrict migration to ages $j \leq 15$, such that migration plays a similar role as fertility in our economic model. This simplifying assumption allows us to treat newborns and immigrants alike. We compute aggregate migration from United Nations (2007) and distribute age-specific migrants in each year equally across all ages $0, \ldots, 15$.

Evaluation

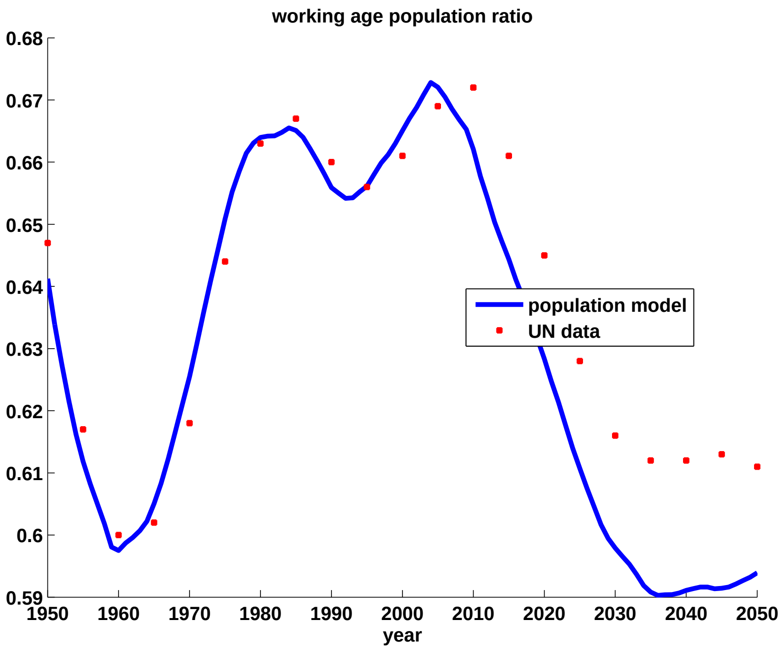
Figures 13-14 display the predicted working-age population and old-age dependency ratios, according to our population model and according to United Nations (2007). Compared to this benchmark, our population model is close to the UN but predicts a slightly stronger decrease of the working-age population ratio and a correspondingly stronger increase of the old-age dependency ratio until 2050.

D Computational Appendix

D.1 Household Problem

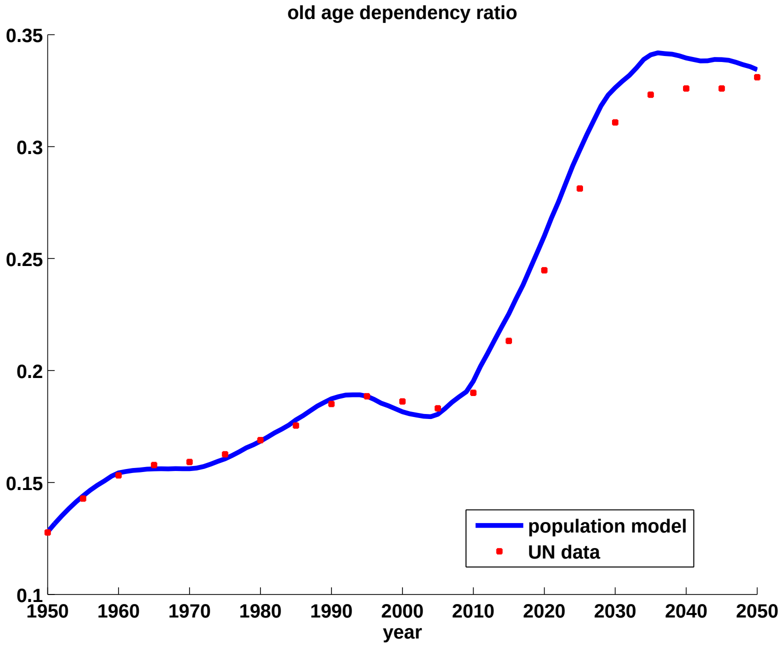
To simplify the description of the solution of the household model for given prices (wage and interest rate), transfers and social-security payments, we focus on steady states and therefore

Figure 13: Working-Age Population Ratio



Notes: Population model: our own predictions of the working-age population ratio based on Human Mortality Database (2008). UN data: working-age population ratio according to United Nations (2007).

Figure 14: Old-Age Dependency Ratio



Notes: Population model: own predictions of the old-age dependency ratio based on Human Mortality Database (2008). UN data: old-age dependency ratio according to United Nations (2007).

drop the time index t. Furthermore, we focus on a de-trended version of the household problem in which consumption c, assets a, wages w and transfers, tr, are transformed into $\tilde{c} = \frac{c}{A}$, $\tilde{a} = \frac{a}{A}$, $\tilde{w} = \frac{w}{A}$ and $\tilde{tr} = \frac{tr}{A}$, where A is the technology level growing at the exogenous rate g.[3] Other variables are not transformed because they are already stationary.

To understand our transformations of the value functions, notice that utility in the last period, period J, takes the form

$$u(c_J, 1 - e_J - \ell_J) = u(c_J, 1) = A^{\phi(1-\sigma)} u(\tilde{c}), \tag{12}$$

Observe that the homotheticity of the utility function is inherited by the value function in period J and in all other periods. We consequently adjust the discount factor to $\tilde{\beta} = \beta\varphi(1+g)^{\phi(1-\sigma)}$.

To understand the transformation of the budget constraint, notice that during the retirement period, the budget constraint is

$$a_{t+1,j+1} = (a_{t,j} + tr_t)(1 + r_t) + \rho_t w_{t+jr-j} \bar{h}_{t+jr-j} \frac{s_{t,j}}{jr - 1} - c_{t,j}. \tag{13}$$

Division by the trend component A then gives

$$\tilde{a}_{t+1,j+1} = \frac{1}{1+g} \left((\tilde{a}_{t,j} + \tilde{tr}_t)(1 + r_t) + \rho_t \tilde{w}_{t+jr-j} (1+g)^{jr-j} \bar{h}_{t+jr-j} \frac{s_{t,j}}{jr - 1} - \tilde{c}_{t,j} \right).$$

[3]These transformations are made for convenience, to simplify the structure of our computer code.

The term $(1+g)^{jr-j}$ reflects the fact that pension income in the US is only indexed to inflation and not to growth of nominal wages.

Taking corresponding adjustments to the budget constraint during the working period, the de-trended version of the household problem is then given by

$$
\begin{aligned}
V(\tilde{a}, h, s, j) = \max_{\tilde{c},\ell,e,\tilde{a}',h',s'} & \left\{ u(\tilde{c}, 1-\ell-e) + \tilde{\beta} V(\tilde{a}', h', s', j+1) \right\} \\
\text{s.t.} & \\
\tilde{a}' &= \frac{1}{1+g} \left((\tilde{a} + \tilde{tr})(1+r) + \tilde{y} - \tilde{c} \right) \\
\tilde{y} &= \begin{cases} \ell h \tilde{w}(1-\tau) & \text{if } j < jr \\ \rho \tilde{w}_{jr} (1+g)^{jr-j} \bar{h}_{jr} \frac{s_{jr}}{jr-1} & \text{if } j \geq jr \end{cases} \\
h' &= g(h, e) && (14) \\
s' &= s + \ell \frac{h}{\bar{h}} && (15) \\
\ell &\in [0,1], \quad e \in [0,1].
\end{aligned}
$$

Here, $g(h, e)$ is the human-capital technology.

Using the budget constraints, now rewrite the above as

$$
\begin{aligned}
V(\tilde{a}, h, s, j) = \max_{\tilde{c},\ell,e,\tilde{a}',h'} & \left\{ u(\tilde{c}, 1-\ell-e) + \tilde{\beta} V\left(\frac{1}{1+g} \left((\tilde{a} + \tilde{tr})(1+r) + \tilde{y} - \tilde{c} \right), g(h,e), s + \ell \frac{h}{\bar{h}}, j+1 \right) \right\} \\
\text{s.t.} & \\
\ell \geq 0. &
\end{aligned}
$$

In the above, we have also replaced the bounded support of time invested and leisure with a one-side constraint on ℓ because the upper constraints, $\ell = 1$, respectively $e = 1$, and the lower constraint, $e = 0$, are never binding due to Inada conditions on the utility function and the functional form of human-capital technology (see below). Recall that $\ell = 0$ for $j \geq jr$.

Denoting by μ_ℓ the Lagrange multiplier on the inequality constraint for ℓ, we can write the first-order conditions as

$$
\begin{aligned}
\tilde{c}: \quad & u_{\tilde{c}} - \tilde{\beta} \frac{1}{1+g} V'_{\tilde{a}'}(\cdot) = 0 && \text{(16a)} \\
\ell: \quad & -u_{1-\ell-e} + \tilde{\beta} \left[h \tilde{w}(1-\tau) \frac{1}{1+g} V'_{\tilde{a}'}(\cdot) + V'_{s'}(\cdot) \frac{h}{\bar{h}} \right] + \mu_\ell = 0 && \text{(16b)} \\
e: \quad & -u_{1-\ell-e} + \tilde{\beta} g_e V'_{h'}(\cdot) = 0 && \text{(16c)}
\end{aligned}
$$

and the envelope conditions as

$$\tilde{a}: \quad V_{\tilde{a}}(\cdot) = \tilde{\beta}\frac{1+r}{1+g}V'_{\tilde{a}'}(\cdot) \tag{17a}$$

$$h: \quad V_h(\cdot) = \begin{cases} \tilde{\beta}\left(\ell\tilde{w}(1-\tau)\frac{1}{1+g}V'_{\tilde{a}'}(\cdot) + g_h V'_{h'}(\cdot) + V'_{s'}(\cdot)\ell\frac{1}{\bar{h}}\right) & \text{if } j < jr \\ \tilde{\beta}V'_{h'}(\cdot)g_h & \text{if } j \geq jr \end{cases} \tag{17b}$$

$$s: \quad V_s(\cdot) = \begin{cases} \tilde{\beta}V'_{s'}(\cdot) & \text{if } j < jr \\ \tilde{\beta}\left(V'_{s'}(\cdot) + \rho\tilde{w}_{jr}(1+g)^{jr-j}\bar{h}_{jr}\frac{1}{jr-1}\frac{1}{1+g}V'_{\tilde{a}'}\right) & \text{if } j \geq jr \end{cases} \tag{17c}$$

Note that for the retirement period, i.e., for $j \geq jr$, equations (16b) and (16c) are irrelevant.

From (16a) and (17a) we obtain

$$V_{\tilde{a}} = (1+r)u_{\tilde{c}} \tag{18}$$

and, using the above in (16a), the familiar inter-temporal Euler equation for consumption follows as

$$u_{\tilde{c}} = \tilde{\beta}\frac{1+r}{1+g}u_{\tilde{c}'}. \tag{19}$$

From (16a) and (16b) we get the intra-temporal Euler equation for leisure,

$$u_{1-\ell-e} = u_{\tilde{c}}h\left(\tilde{w}(1-\tau) + (1+g)\frac{V'_{s'}}{V'_{\tilde{a}'}}\frac{1}{\bar{h}}\right) + \mu_\ell. \tag{20}$$

From the human capital technology (3) we further have

$$g_e = \xi\psi(eh)^{\psi-1}h \tag{21a}$$

$$g_h = (1-\delta^h) + \xi\psi(eh)^{\psi-1}e. \tag{21b}$$

We loop backwards on j from $j = J-1, \ldots, 1$ by taking an initial guess of $[\tilde{c}_J, h_J]$ as given and by initializing $V_{\tilde{a}'}(\cdot, J) = V_{h'}(\cdot, J) = V_{s'}(\cdot, J) = 0$. During retirement, that is, for all ages $j \geq jr$, our solution procedure is standard backward shooting using the first-order conditions. However, during the period of human-capital formation, that is, for all ages $j < jr$, the first-order conditions would not be sufficient if the problem is not a convex-programming problem. Thus, our backward-shooting algorithm will not necessarily find the true solution. In fact, this may be the case in human-capital models such as ours because the effective wage rate is endogenous (it depends on the human-capital investment decision). For a given initial guess $[\tilde{c}_J, h_J]$, we therefore first compute a solution and then consider variations of initial guesses of $[\tilde{c}_J, h_J]$ on a large grid and check whether we converge to the same unique solution. In all of our scenarios, we never find any multiplicities. The details of our steps are as follows:

1. In each j, $h_{j+1}, V_{\tilde{a}'}(\cdot, j+1), V_{h'}(\cdot, j+1)$, and $V_{s'}(\cdot, j+1)$ are known.

2. Compute $u_{\tilde{c}}$ from (16a).

3. For $j \geq jr$, compute h_j from (3) by setting $e_j = \ell_j = 0$ and by taking h_{j+1} as given. Compute $\tilde{c}_j$ directly from Equation (24) below.

4. For $j < jr$:

   (a) Guess h_j

   (b) Compute e_j from (3) as

   $$e_j \quad = \tfrac{1}{h_j} \left(\tfrac{h_{j+1} - h_j(1-\delta^h)}{\xi} \right)^{\frac{1}{\psi}}. \tag{22}$$

   (c) Compute $lcr_j = \frac{1-e_j-\ell_j}{\tilde{c}_j}$, the leisure-to- consumption ratio, from (20), as follows: From our functional-form assumption on utility, marginal utilities are given by

   $$\begin{aligned} u_{\tilde{c}} &= \left(\tilde{c}^{\phi}(1-\ell-e)^{1-\phi}\right)^{-\sigma} \phi \tilde{c}^{\phi-1}(1-\ell-e)^{1-\phi} \\ u_{1-\ell-e} &= \left(\tilde{c}^{\phi}(1-\ell-e)^{1-\phi}\right)^{-\sigma} (1-\phi)\tilde{c}^{\phi}(1-\ell-e)^{-\phi} \end{aligned}$$

   Hence, we obtain from (20) the familiar equation:

   $$\frac{u_{1-\ell-e}}{u_{\tilde{c}}} = h\left(\tilde{w}(1-\tau) + (1+g)\frac{V'_{s'}}{V'_{\tilde{a}'}}\frac{1}{\bar{h}}\right) = \frac{1-\phi}{\phi}\frac{\tilde{c}}{1-\ell-e},$$

   and therefore:

   $$lcr_j = \frac{1-e_j-\ell_j}{\tilde{c}_j} = \frac{1-\phi}{\phi}\left(h\left[\tilde{w}(1-\tau) + (1+g)\frac{V'_{s'}}{V'_{\tilde{a}'}}\frac{1}{\bar{h}}\right]\right)^{-1}. \tag{23}$$

   (d) Next, compute $\tilde{c}_j$ as follows. Notice first that one may also write marginal utility from consumption as

   $$u_{\tilde{c}} = \phi \tilde{c}^{\phi(1-\sigma)-1}(1-\ell-e)^{(1-\sigma)(1-\phi)}. \tag{24}$$

   Using (23) in (24), we then obtain

   $$\begin{aligned} u_{\tilde{c}} &= \phi \tilde{c}^{\phi(1-\sigma)-1}(lcr \cdot \tilde{c})^{(1-\sigma)(1-\phi)} \\ &= \phi \tilde{c}^{-\sigma} \cdot lcr^{(1-\sigma)(1-\phi)}. \end{aligned} \tag{25}$$

   Because $u_{\tilde{c}}$ is given from (16a), we can now compute $\tilde{c}$ as

   $$\tilde{c}_j = \left(\frac{u_{\tilde{c}_j}}{\phi \cdot lcr_j^{(1-\sigma)(1-\phi)}}\right)^{-\frac{1}{\sigma}}. \tag{26}$$

   (e) Given $\tilde{c}_j$, e_j, compute labor, ℓ_j, as

   $$\ell_j = 1 - lcr_j \cdot \tilde{c}_j - e_j.$$

(f) If $\ell_j < 0$, set $\ell_j = 0$ and recompute $\tilde{c}_j$ from (24) as

$$\tilde{c} = \left(\frac{u_{\tilde{c}}}{\phi(1-e)^{(1-\sigma)(1-\phi)}} \right)^{\frac{1}{\phi(1-\sigma)-1}}.$$

(g) Finally, use (21a) in (16c) and define the resulting equation as a distance function $f(h)$. We solve for the root of f to obtain h_j by a non-linear solver iterating steps 4a through 4g until convergence. The following proposition establishes that this solution is unique.

Proposition 1. *For given values of human capital next period, h_{j+1}, and marginal values next period, $V'_{\tilde{a}'}$, $V'_{h'}$ and $V'_{s'}$, a solution h_j to the first-order conditions (16a), (16b), (16c), and the human-capital constraint (3) exists and is unique.*

We present the proof of Proposition 1 after the description of our algorithm.

5. Update as follows:

   (a) Update $V_{\tilde{a}}$ using either (17a) or (18).

   (b) Update V_h using (17b).

   (c) Update V_s using (17c).

Next, loop forward on the human-capital technology (3) for given h_0 and $\{e_j\}_{j=1}^J$ to compute an update of h_J denoted by h_J^n. Compute the present discounted value of consumption, PVC, and, using the previously computed values $\{h_j^n\}_{j=1}^J$, $\{\ell_j^n\}_{j=1}^J$, and $\{p_j^n\}_{j=jr}^J$ compute the present discounted value of income, PVI. Use the relationship

$$\tilde{c}_0^n = \tilde{c}_0 \cdot \frac{PVI}{PVC} \tag{27}$$

to form an update of initial consumption, $\tilde{c}_0^n$, and next use the Euler equations for consumption to form an update of $\tilde{c}_J$, denoted as $\tilde{c}_J^n$. Define the distance functions

$$g_1(\tilde{c}_J, h_J) = \tilde{c}_J - \tilde{c}_J^n \tag{28a}$$

$$g_2(\tilde{c}_J, h_J) = h_J - h_J^n. \tag{28b}$$

In our search for general-equilibrium prices, constraints of the household model are occasionally binding. Therefore, solution of the system of equations in (28) using Newton-based methods, e.g., Broyden's method, is instable. We solve this problem by a nested Brent algorithm, that is, we solve two nested univariate problems, an outer one for $\tilde{c}_J$ and an inner one for h_J.

Check for uniqueness: Observe that our nested Brent algorithm assumes that the functions in (28) exhibit a unique root. What is computed above is a candidate solution under the assumption that the first-order conditions are necessary and sufficient. As a consequence of potential non-convexities of our programming problem, first-order conditions may, however, not be sufficient, and our procedure may therefore not give the unique global optimum. To systematically check whether we also always converge to the unique optimum, we fix, after convergence of the household problem, a large box around the previously computed $[\tilde{c}_J, h_J]$. Precisely, we choose as boundaries for this box $\pm 50\%$ of the solutions in the respective dimensions. For these alternative starting values, we then check whether there is an additional solution to the system of equations (28). For all of these combinations, our procedure always converged, and we never detected any such multiplicities.

Proof of Proposition 1. Consider the cases with and without a binding constraint on labor supply separately.

1. Consider an interior solution for labor supply, i.e., $\ell_j > 0$ and $\mu_\ell = 0$. In this case, one can find the values of e_j and h_j satisfying the first-order conditions independently of c_j and l_j. Combining the first-order conditions for labor supply (16b) and human-capital investment (16c) yields

$$e_j h_j = \left(\frac{\xi\psi(1+g)V_{h'}(\cdot)}{\tilde{w}(1-\tau)V_{\tilde{a}'}(\cdot) + (1+g)V'_{s'}\frac{1}{h}} \right)^{\frac{1}{1-\psi}}. \tag{29}$$

   Note that the term on the right-hand side does not depend on h_j. Finally, substituting $e_j h_j$ in Equation (3) gives

$$h_{j+1} = h_j(1-\delta^h) + \xi \left(\frac{\xi\psi(1+g)V_{h'}(\cdot)}{\tilde{w}(1-\tau)V_{\tilde{a}'}(\cdot) + (1+g)V'_{s'}\frac{1}{h}} \right)^{\frac{\psi}{1-\psi}}. \tag{30}$$

   Clearly, this equation has a unique solution for h_j. Given this value for h_j, Equation (29) determines a unique value for e_j.

2. Consider a binding constraint on labor supply, i.e., $\ell_j = 0$ and $\mu_\ell > 0$. In this case, the values of e_j and h_j satisfying the first-order conditions depend on c_j. The first-order condition for human-capital investment (16c) reads as

$$(1-\phi)\tilde{c}^{\phi(1-\sigma)}(1-e)^{(1-\phi)(1-\sigma)-1} = \tilde{\beta}\xi\psi e^{\psi-1}h^{\psi}V'_{h'} \tag{31}$$

   , and the first-order condition for consumption is given by

$$\phi\tilde{c}^{\phi(1-\sigma)-1}(1-e)^{(1-\phi)(1-\sigma)} = \tilde{\beta}\frac{1}{1+g}V'_{\tilde{a}'}. \tag{32}$$

   Combining these two equations to eliminate c yields

$$h = e^{\frac{1-\psi}{\psi}}(1-e)^{\frac{1}{\psi}\frac{\sigma}{\phi(1-\sigma)-1}}\Phi^{\frac{1}{\psi}}, \tag{33}$$

   where

$$\Phi = (1-\phi)\left(\phi\tilde{\beta}\frac{1}{1+g}V'_{\tilde{a}'}\right)^{\frac{\phi(1-\sigma)}{\phi(1-\sigma)-1}}(\tilde{\beta}\xi\psi V'_{h'})^{-1}. \tag{34}$$

   The (e,h) combination we are seeking needs to satisfy Equation (33) and the human-capital constraint (3). This means we have a system of two equations in the two unknowns, e and h. Because both equations (33) and (3) are continuous, and the admissible values of e are in the range $0 \le e \le 1$,

- existence of a solution follows from the fact that in Equation (33), $h = 0$, if $e = 0$, and $h \to \infty$, if $e \to 1$ because $\phi(1-\sigma) < 1$; and in Equation (3), $h = h'/(1-\delta^h) > 0$, if $e = 0$, and h is finite if $e = 1$.
- uniqueness of the solution to these two equations follows because in the relevant range of e, $\frac{\partial h}{\partial e} < 0$ in Equation (3) by the implicit-function theorem, and $\frac{\partial h}{\partial e} > 0$ in Equation (33) because the derivative of h w.r.t. e in Equation (33) is

$$\frac{\partial h}{\partial e} = \left(\frac{1-\psi}{\psi} e^{\frac{1-\psi}{\psi}-1} (1-e)^{\frac{1}{\psi}\frac{\sigma}{\phi(1-\sigma)-1}} - \frac{1}{\psi}\frac{\sigma}{\phi(1-\sigma)-1} e^{\frac{1-\psi}{\psi}-1} (1-e)^{\frac{1}{\psi}\frac{\sigma}{\phi(1-\sigma)-1}-1} \right) \Phi^{\frac{1}{\psi}},$$

,

and the second term in brackets is positive for $\phi(1-\sigma) < 1$.

□

D.2 The Aggregate Model

For a given $r \times 1$ vector $\vec{\Psi}$ of structural model parameters, we first solve for an "artificial" initial steady state in period $t = 0$, which gives initial distributions of assets and human capital. We thereby presume that households assume prices to remain constant for all periods $t \in \{0, \ldots, T\}$ and are then surprised by the actual price changes induced by the transitional dynamics. Next, we solve for the final steady state of our model, which is reached in period T and supported by our demographic projections (see Appendix C). For both steady states, we solve for the equilibrium of the aggregate model by iterating on the $m \times 1$ steady-state vector $\vec{P}_{ss} = [p_1, \ldots, p_m]'$. In our case, $m = 4$. p_1 is the capital intensity, p_2 are transfers (as a fraction of wages), p_3 are social-security contribution (or replacement) rates, and p_4 is the average (hours weighted) human-capital stock. Notice that all elements of $\vec{P}_{ss}$ are constant in the steady state.

The solution for the respective initial and final steady states of the model involves the following steps:

1. In iteration q for a guess of $\vec{P}^q_{ss}$ solve the household problem.
2. Update variables in $\vec{P}_{ss}$ as follows:
   (a) Aggregate across households to obtain aggregate assets and aggregate labor supply to form an update of the capital intensity, p_1^n.
   (b) Calculate an update of bequests to get p_2^n.
   (c) Using the update of labor supply, update social-security contribution (or replacement) rates to get p_3^n.
   (d) Use labor supply and human-capital decisions to form an update of the average human-capital stock, p_4^n.
3. Collect the updated variables in $\vec{P}^n_{ss}$ and notice that $\vec{P}^n_{ss} = H(\vec{P}_{ss})$ where H is a vector-valued non-linear function.

4. Define the root-finding problem $G(\vec{P}_{ss}) = \vec{P}_{ss} - H(\vec{P}_{ss})$, and iterate on $\vec{P}_{ss}$ until convergence. We use Broyden's method to solve the problem and denote the final approximate Jacobi matrix by B_{ss}.

Next, we solve for the transitional dynamics by the following steps:

1. Use the steady-state solutions to form a non-linear interpolation to obtain the starting values for the $m(T-2) \times 1$ vector of equilibrium prices, $\vec{P} = [\vec{p}_1', \ldots, \vec{p}_m']'$, where $p_i, i = 1, \ldots, m$ are vectors of length $(T-2) \times 1$.

2. In iteration q for guess $\vec{P}^q$, solve the household problem. We do so by iterating backwards in time for $t = T-1, \ldots, 2$ to obtain the decision rules and forward for $t = 2, \ldots, T-1$ for aggregation.

3. Update variables as in the steady-state solutions, and denote by $\tilde{\vec{P}} = H(\vec{P})$ the $m(T-2) \times 1$ vector of updated variables.

4. Define the root-finding problem as $G(\vec{P}) = \vec{P} - H(\vec{P})$. Because T is large, this problem is substantially larger than the steady-state root-finding problem, and we use the Gauss-Seidel-Quasi-Newton algorithm suggested in Ludwig (2007) to form and update guesses of an approximate Jacobi matrix of the system of $m(T-2)$ non-linear equations. We initialize these loops with a scaled-up version of B_{ss}.

D.3 Calibration of Structural Model Parameters

We split the $r \times 1$ vector of structural model parameters, $\vec{\Psi}$, as $\vec{\Psi} = \left[(\vec{\Psi}^e)', (\vec{\Psi}^f)'\right]'$. $\vec{\Psi}^f$ is a vector of predetermined (fixed) parameters, whereas the $e \times 1$ vector $\vec{\Psi}^e$ is estimated by minimum distance (unconditional matching of moments using e moment conditions). Denote by

$$u_t(\vec{\Psi}^e) = y_t - f(\vec{\Psi}^e) \text{ for } t = 0, \ldots, T_0 \tag{35}$$

the GMM error as the distance between actual values, y_t, and model-simulated (predicted) values, $f(\vec{\Psi}^e)$.

Under the assumption that the model is correctly specified, the restrictions on the GMM error can be written as

$$E[u_t(\vec{\Psi}_0^e)] = 0, \tag{36}$$

where $\tilde{\Psi}_0^e$ denotes the vector of true values. Denote sample averages of u_t as

$$g_{T_0}(\vec{\Psi}^e) \equiv \frac{1}{T_0 + 1} \sum_{t=0}^{T_0} u_t(\vec{\Psi}^e). \tag{37}$$

We estimate the elements of $\vec{\Psi}^e$ by setting these sample averages to zero (up to some tolerance level).

In our economic model, only two parameters are pre-determined, and we therefore have

$$\vec{\Psi}^f = [\sigma, h_0]'. \tag{38}$$

The vector $\vec{\Psi}^e$ is given by

$$\vec{\Psi}^e = \left[g, \alpha, \delta, \beta, \phi, \psi, \xi, \delta^h\right]'. \tag{39}$$

We estimate the structural model parameters using data from various sources for the period 1960, ..., 2004. Hence $T_0 = 44$. The parameters $\vec{\Psi}_1^e = [g, \alpha]'$ are directly determined using NIPA data on GDP, fixed assets, wages and labor supply. The remaining structural model parameters, $\vec{\Psi}_2^e = [\delta, \beta, \phi, \psi, \xi, \delta^h]'$ are estimated by simulation. Our calibration targets are summarized in Table 7.

Table 7: Calibration Targets

Parameter	Target	Moment
$\vec{\Psi}^f$		
σ	predetermined parameter	
h_0	predetermined parameter	
$\vec{\Psi}_1^e$		
g^A	growth rate of Solow residual	0.018
α	share of wage income	0.33
$\vec{\Psi}_2^e$		
δ	investment output ratio	0.2
β	capital output ratio	2.8
ϕ	average hours worked	0.33
ψ, ξ, δ^h	coefficients of wage polynomial (from PSID)	

Determining the subset of parameters $\vec{\Psi}_2^e$ along the transition is a computationally complex problem that we translate into an equivalent simple problem. The point of departure of our procedure is the insight that calibrating the model for a steady state is easy and fast. However, simulated steady-state moments may differ quite substantially from simulated averages along the transition, even when the steady state is chosen to lie in the middle of the calibration period, in our case, year 1980. We therefore proceed as follows.

1. Initialization: Choose a vector of scaling factors, $\vec{sf}$, of length e_2 that appropriately scales the steady-state calibration targets (see below).

2. Calibrate the model in some steady-state year, e.g., 1980, by solving the system of equations

$$\frac{\bar{y}_{2,i}^e}{sf_i} - f_{2,i}^{e,ss}(\vec{\Psi}) \tag{40}$$

   for all $i = 1, \ldots, e_2$ to get $\hat{\vec{\Psi}}_2^e$. Here, $\bar{y}_{2,i}^e$ is the average of moment i in the data for the calibration period (1960-2004), e.g., the investment-output ratio for $i = 1$.

3. For the estimated parameter vector, $\hat{\vec{\Psi}}_2^e$, solve the model along the transition.

4. Compute the relevant simulated moments for the transition, $f_2^e(\vec{\Psi})$.

5. Update the vector of scaling vectors as

$$sf_i = \frac{f^e_{2,i}(\vec{\Psi})}{f^{e,ss}_{2,i}(\vec{\Psi})} \tag{41}$$

for all $i = 1, \ldots, e_2$.

6. Continue with step 2 until convergence on scaling factors (fixed-point problem).

We thereby translate a complex root-finding problem into a combination of a simple root-finding problem (steady- state calibration) and a fixed-point iteration on scaling factors. Because scaling factors are relatively insensitive to Ψ^e_2, convergence is fast and robust. The resulting scaling factors range from 0.94 to 1.29, which means that differences between simulated moments in the artificial steady-state year (1980) and averages during the transition are large (up to 30%). This also implies that calibrating the model in some artificial steady-state year would only lead to significantly biased estimates of structural model parameters.

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